The Precalculus Student's Guide To The Universe

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The Precalculus Student's Guide To The Universe

For Dr. Mahmoud Zeinalian

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The Precalculus Student's Guide To The Universe
Introduction

Pre-calculus is often presented as a slightly more advanced version of a good algebra II course, rather than a course which actually prepares students for and motivates the study of calculus. Topics such as trigonometry, rational functions and logarithms are revisited but not expanded upon in a way which is inspiring to strong math students ready for new ideas. Additional topics such as sequences and matrices are introduced, but no attempt is made to connect them to the rest of the course content. In the same vein, early parts of calculus such as limits are tossed in at the end. As a result, the course tends to have a scattered and disconnected feel, and the students have little sense of why calculus (and higher mathematics in general) may be relevant for careers they are interested in.

As a teacher at the Masters School in Dobbs Ferry, New York, I chose a new approach to teaching the honors-level pre-calculus course. I tied the content together with central themes of calculus – limits, rates of change and accumulation. I gave special emphasis to concepts such as increments, difference equations and sequences of partial sums, as they are the discrete analogues of derivatives, differential equations and integrals. Having the opportunity to explore the relationships between change and accumulation in the discrete setting helped students understand the critical role these ideas play in mathematical modeling and provided motivation for the study of calculus. Because of the discrete nature of computer data, understanding the interconnections between discrete and continuous models is more essential now than ever before. Lastly, I dedicated a significant portion of the course to applications in probability theory, a fundamental ingredient of virtually all applied mathematics (and a rich source of practical problems which require calculus to solve). Students who completed the course had a stronger sense of what calculus is all about than with previous approaches, and they were better-prepared to grasp the subtleties that AP Calculus® students tend to struggle with most.

The content of the course is divided into five large units, designed to be completed in sequence. Each lesson within a unit could take two to three class sessions to complete, depending on students' backgrounds and abilities as well as scheduling. New concepts are introduced with detailed explanations, lots of worked examples and intermittent exercises which assess basic understanding. The "Problems" section at the end of each lesson contains a wide variety of questions which tend to fall into one of the following three categories:

I. Standard problems which resemble the worked examples and exercises from the lesson,

II. Critical thinking problems which test for a deeper level of understanding of the content by challenging students to apply their knowledge in unfamiliar contexts,

III. Scaffolding questions which combine ideas from previous units with material from the current lesson.

Due to the abundance of problems of types II and III, the book could also be used as the basis for a problem-based learning course (especially for a strong group of students who do not need much review

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of algebra II concepts). Each unit concludes with a "Further Explorations" section containing guided projects which extend the presented material. The purpose of the projects is to enable students to explore the practical value of the powerful mathematical tools they have developed, and to help students understand the types of problems which cannot be solved without more machinery (from calculus and beyond). They are suitable for in-class "labs" or independent study. Several of the Further Explorations sections require the use of graphing or spreadsheet software. Recommendations for popular free options such as Desmos.com and Google Spreadsheets are provided.

Below is a more detailed breakdown of some of the unique aspects of this curriculum:

Unit 1 provides a fast-paced review of the key skills and concepts students are expected to have mastered in earlier algebra courses. During this unit, students will also learn how to write a formal proof of an algebraic identity, a valuable skill which is necessary for the rest of the course as well as more advanced mathematics courses. Several other important ideas which reappear throughout the text, like binary operations on sets and general solutions, are also introduced.

Unit 2 is a study of sequences and series. The presentation is more involved than that of a typical pre-calculus course. More emphasis is placed on the relationship between increments, difference equations and sequences of partial sums. When students move on to calculus, they will see these concepts replaced by derivatives, differential equations and integrals in the continuous setting. Having the opportunity to explore the discrete case in depth sheds light on the practical value of studying calculus. The last two lessons in this unit provide a fairly rigorous introduction to limits in the context of sequences. Since limits form the basis of all the tools of calculus, it is important for students to have the opportunity to fully grasp that the abstract and fuzzy-sounding notion of a limit is not only mathematically precise but profoundly useful. Introducing limits in the discrete context first helps students ease into this valuable but subtle new concept gradually.

Unit 3 is a survey of topics in combinatorics and discrete probability. In the first lesson, basic tools like the fundamental counting principle, factorials, permutations, combinations and the binomial formula are presented. As in previous units, students are expected to complete algebraic proofs of identities in addition to solving typical application problems. The subsequent lessons cover the foundations of probability, conditioning and random variables, including detailed discussions of uniform, binomial, geometric, hypergeometric and Poisson probability. Special effort is made to show how concepts from Unit 2 like summations, partial sums and limits are particularly useful in the context of discrete probability.

Unit 4 is an in-depth study of functions and their properties. Though fundamental concepts are reviewed, there is an expectation that students have considerable experience working with functions and are already familiar with basic skills and concepts such as graphing, inverse functions and transformations. Instead, emphasis is placed on more advanced topics such as continuity, end behavior, function identities and binary operations on sets of functions. As with the rest of the text, a major goal of this unit is for students to become comfortable with a higher level of abstraction. For many problems, students are asked to answer questions about an arbitrary or unknown function which is assumed to have certain given properties. These types of problems help open the students' mathematical imaginations and help prepare them for more advanced courses such as AP Calculus.

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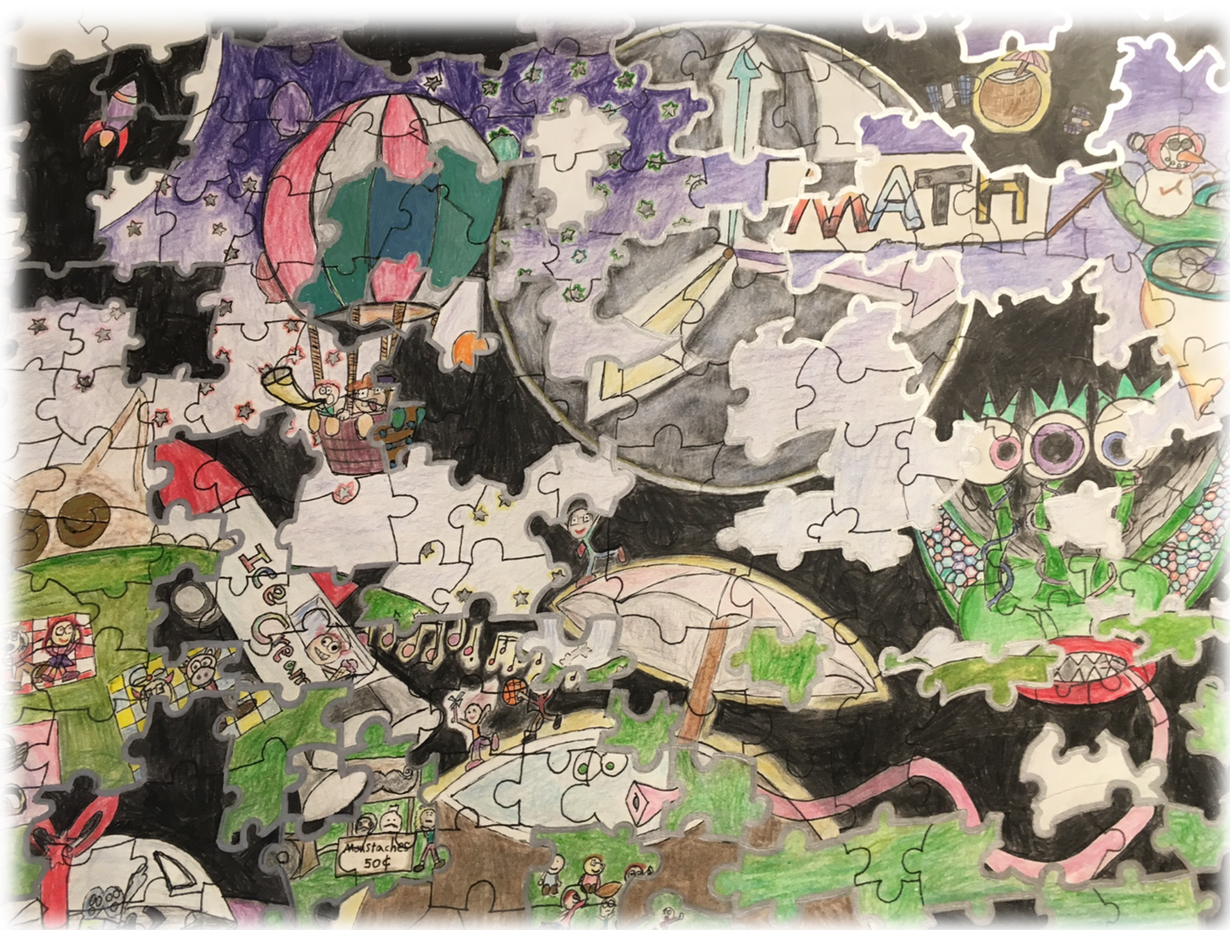
In Unit 5, students learn to apply the tools and reasoning skills introduced in previous units (especially Unit 4) to analyze trigonometric and complex functions. Again, the fundamentals of angles, trigonometry and complex numbers are all reviewed, but there is an expectation that students have already been exposed to the basics in previous courses and are ready for a more advanced treatment. Some sophisticated topics such as Fourier series, Euler's identity and conformal mappings are presented on a very elementary level in order to help motivate the study of calculus and higher mathematics.

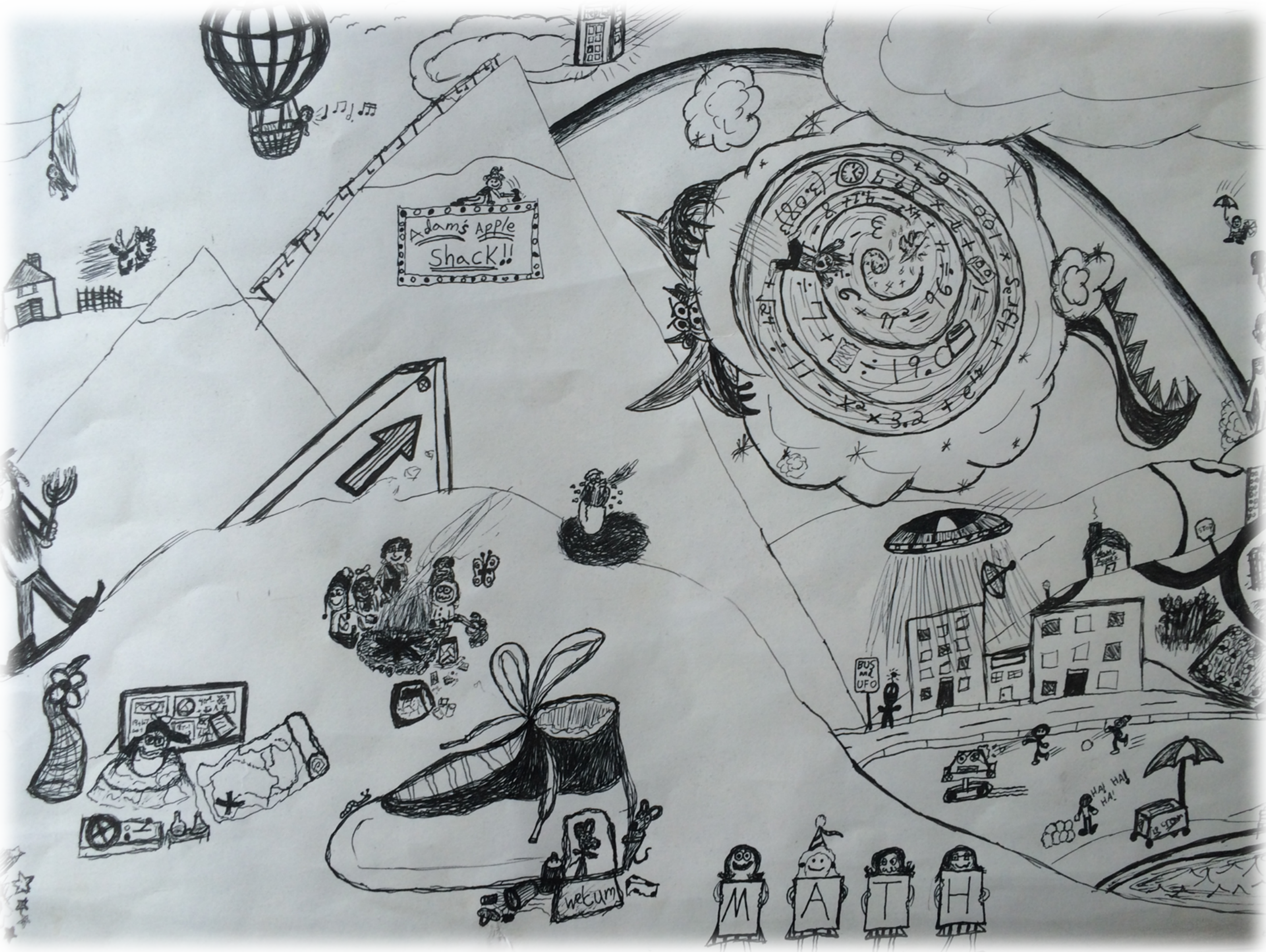
I would like to give a special thank you to everyone who supported me to develop and implement the material found in this book. In particular, I am forever indebted to the entire Masters School Mathematics Department for helping me grow as an educator and for constantly encouraging me to experiment with new ideas. They are my greatest teacher role models and the voices in my head to this day when I teach. I would also like to thank Alan Flyer and Cheryl Kornfeld at the Mathnasium of Roslyn, New York for first inspiring me to start this project when I was in college (*and* for inspiring me to create the gigantic doodles found on the cover and elsewhere in the book between working with students!). Furthermore, I am so grateful to my parents for their constant support throughout the entire writing process, and to my uncle Joe for helping me prepare the book for publication.

Finally, I would like to thank all of my students for putting up with my ridiculous personality, being open to trying new things and offering helpful feedback. I hope the time we spent together in class had some meaning for them (as it did for me).



Unit 1
Algebra Smackdown

The purpose of this short unit is to review all the basic skills and concepts from algebra we will need for the rest of this course. The last lesson introduces the idea of a *general solution* – that is, a solution to an entire *class* of problems. Enjoy!



Unit 1 Key Terms And Skills

Lesson 1.1:

Terms: set, element, subset, complement, intersection, union, cardinality (for finite sets), empty set (ϕ), natural numbers ($\mathbb{N}$), integers ($\mathbb{Z}$), rational numbers ($\mathbb{Q}$), real numbers ($\mathbb{R}$), open/closed interval, binary operation, closure (of a set under a binary operation), pi (π), Euler's constant (e)

Skills:

a. Interpret and write simple statements in set notation

b. Convert between interval notation and set-builder notation

c. Determine whether or not a number belongs to a certain set

d. Determine whether or not a set of numbers is closed under a binary operation

Lesson 1.2:

Terms: variable, constant, algebraic expression, implied domain (of an algebraic expression), equivalence (of algebraic expressions), algebraic identity, exponent, logarithm, greatest-/least-integer operator

Skills:

a. Evaluate an algebraic expression for given values of the variable(s)

b. Specify the implied domain of an algebraic expression

c. Determine whether or not two algebraic expressions are equivalent

d. **Prove** that two algebraic expressions are equivalent using the distributive property, factoring methods, properties of fractions and laws of logs and exponents

e. Provide a counterexample which proves two algebraic expressions are *not* equivalent

f. Estimate the value of an expression involving exponents and logarithms

Unit 1 Key Terms And Skills (Cont.)

Lesson 1.3:

Terms: equation, inequality, solution, solution set, equivalent equations/inequalities, literal equation/inequality

Skills:

a. Determine whether a given value is a solution to an equation or inequality

b. Solve equations and inequalities studied in Algebra II (except trig equations); see Prerequisite Skills And Concepts

c. Write the solution set of an equation or inequality using both set notation and interval notation

d. Isolate a variable in a literal equation or inequality

Lesson 1.4:

Terms: general solution, parameter/constant, "reality check"

Skills:

a. Find the general solution to a literal equation or inequality associated with a word problem

b. Determine cases (parameter values) in which a solution exists or does not exist

c. Complete a "reality check" for a general solution and interpret the results in context